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Moving Embedded Solitons

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The first theoretical results are reported predicting *moving* isolated solitons residing inside (*embedded* into) the continuous spectrum of radiation modes. The model taken is a Bragg-grating medium with Kerr nonlinearity and additional second derivative (wave) terms. In parameter space, moving embedded solitons (ESs) are doubly isolated (codimension 2). Unlike the quiescent branches, moving ES existence depends sensitively on the ratio of self/cross-phase modulation, with only one solution occurring at the physical value $\sigma = 1/2$. Like quiescent ESs, moving ESs are argued to be stable to linear approximation, and *semi*-stable nonlinearly. This tri-stable configuration (static, left and right-travelling) has potential for optical storage of information. Estimates suggest moving ESs may be experimentally observed as ~ 10 fs pulses with velocity $\leq 1/10$ th that of light.

Recent studies have revealed a novel type of *soliton* (“solitary wave” is more accurate since we do not assume integrability) that is *embedded* into the continuous spectrum, i.e., the soliton’s internal frequency is in resonance with linear (radiation) waves. Generally, such a soliton should not exist, one finding instead a “quasi-soliton” with non-vanishing oscillatory tails (radiation component) [1]. Nevertheless, *bona fide* (exponentially decaying) solitons can exist as *codimension-one* solutions if, at discrete values of the (quasi-)soliton’s internal frequency, the amplitude of the tail exactly vanishes, while the soliton remains embedded into the continuous spectrum. This requires the spectrum of the corresponding linearized system to consist of (at least) two branches, one corresponding to exponentially localized solutions, and the other to radiation modes. In terms of the travelling-wave ordinary differential equations (ODEs), the origin must be a *saddle-centre* equilibrium.

Examples of such *embedded solitons* (ESs) were found in water-wave models [2,3] and in several nonlinear-optical ones, e.g., a Bragg grating with dispersion and/or diffraction terms [4], and second-harmonic generation (SHG) in the presence of a self-defocusing Kerr nonlinearity ([5,6]). The term “ES” was proposed in Ref. [5].

ESs are interesting for several reasons, firstly because they frequently appear when higher-order (singular) perturbations are added to the system, which may completely change its soliton spectrum (see e.g. [4]). Secondly, optical ESs may have considerable potential for applications, just because they are isolated solitons rather than members of continuous families. Finally, and most crucially for their physical applications, it appears that ESs are *semi*-stable objects. That is, as is proven in Ref. [5] analytically in a general form applicable to

ESs in any system, and checked numerically in terms of the SHG - defocusing-Kerr model, they are stable in the linear approximation but are subject to a slowly growing (sub-exponential) *one-sided* nonlinear instability (see below).

A crucial remaining issue for applications is whether *moving* ESs (ones with non-zero momentum) may occur in systems where they cannot be generated by a straightforward transformation (the absence of such a transformation is typical in nonlinear-optical systems). The objective of the present work is to search for moving ESs in a physically important system, viz., a nonlinear Bragg-grating model similar to that which was introduced in [4], which takes into account second-derivative (wave) terms. In fact (see below), this system has a broader physical purport than was originally assumed in [4]. The absence of Galilean or Lorentzian invariance in it is obvious because there is a reference frame in which the Bragg grating is quiescent. Although exact solutions for moving solitons are available in the traditional version of this model which neglects the second derivative terms [7,8], they can be obtained by the Lorentz transformation from the quiescent solitons only in the limiting case of the Thirring model [9], which is completely integrable [10].

We start from a system of partial differential equations (PDEs) governing evolution of right- ($u(x, t)$) and left- ($v(x, t)$) traveling waves that continuously transform into each other due to the resonant reflection on the grating:

$$\begin{aligned} iu_t + iu_x + (2k)^{-1}(u_{xx} - u_{tt}) + (\sigma|u|^2 + |v|^2)u + v &= 0, \\ iv_t - iv_x + (2k)^{-1}(v_{xx} - v_{tt}) + (\sigma|v|^2 + |u|^2)v + u &= 0. \end{aligned}$$

Here, the cubic and linear cross-coupling terms account,

respectively, for nonlinear cross-phase modulations and Bragg scattering. The most natural physical value of the relative self-modulation coefficient σ is $1/2$, but it will be quite useful to keep it as an arbitrary positive parameter. Note that Eqs. (1) and (2) have three natural integrals of motion: the energy (norm) and momentum,

$$E \equiv \int_{-\infty}^{+\infty} [|u(x)|^2 + |v(x)|^2] dx, \quad P \equiv i \int_{-\infty}^{+\infty} (u_x^* u + v_x^* v) dx$$

and the Hamiltonian, an expression for which is obvious.

The energy plays a crucial role in analyzing ES stability [5], as ESs are isolated solutions with uniquely determined values of the energy. Hence, any small perturbation which slightly increases an ES's energy, is safe, while a perturbation that slightly decreases energy triggers a slow (sub-exponential) decay into radiation. So in this sense, the weak instability of an ES is one-sided, as mentioned above.

Eqs. (1) and (2) can be derived from Maxwell's equations for a nonlinear medium, assuming a superposition of two counter-propagating electromagnetic waves, $u(x, t) \exp(ikx - i\omega t)$ and $v(x, t) \exp(-ikx - i\omega t)$, where the wavenumber k and frequency ω are related by the dispersion relation for linear waves (disregarding the Bragg coupling between them), and the amplitude functions $u(x, t)$ and $v(x, t)$ are slowly varying compared with the rapidly oscillating carrier waves. Taking for simplicity a medium whose temporal dispersion may be neglected, and setting the group velocity of electromagnetic waves $c_0 \equiv 1$ (hence, $\omega = k$), one derives, to lowest order in the small parameter $1/2k$, Eqs. (1) and (2) without second-derivative terms. This is the standard model of a Bragg reflector filled by a Kerr-nonlinear medium [7,8]. At the next order, the second derivatives (wave terms) in Eqs. (1) and (2) must be kept. As shown in [4], these terms, even if they appear with a small coefficient, drastically alter the soliton spectrum of the model (which might be expected, as this is a *singular perturbation*, that increases the order of the PDE system). In an experiment, the effect of the additional terms may be seen if the observation time is large enough. An estimate of relevant physical parameters will be given below.

Solitons are solutions of the form

$$u(x, t) = \exp(-i\Delta\omega t) U(\xi), \quad v(x, t) = \exp(-i\Delta\omega t) V(\xi),$$

where $\xi \equiv x - vt$, v is the soliton's velocity, and $\Delta\omega$ is its frequency shift. The substitution of this expression into Eqs. (1) and (2) yields the ODEs,

$$\chi U + i(1 - c)U' + DU'' + (\sigma|U|^2 + |V|^2)U + V = 0, \quad (1)$$

$$\chi V - i(1 + c)V' + DV'' + (\sigma|V|^2 + |U|^2)V + U = 0, \quad (2)$$

where $\chi \equiv \Delta\omega + (\Delta\omega)^2/2k$, the effective velocity is $c \equiv (1 + \Delta\omega/k)v$, and an effective dispersion coefficient is $D \equiv (1 - v^2)/2k$.

The same ODEs were derived in [4] in two different physical contexts: (i) a nonlinear Bragg-grating medium in which *spatial dispersion* is taken into regard, and (ii) spatial evolution in a planar waveguide equipped with Bragg grating in the form of a set of parallel scores, in which ordinary diffraction is taken into regard, the temporal variable t being interpreted as a propagation coordinate z . While all these systems are described by the equations (1) and (2), the new physical interpretation, viz., a usual Bragg-grating system with wave terms taken into regard, seems most fundamental.

To look for ES solutions, we must first satisfy the necessary condition, viz., that the linearization of the ODEs should be of saddle-centre type. That is, at least one pair of eigenvalues must be purely imaginary (otherwise, we are dealing with *regular*, i.e. non-embedded, solitons), and at least one pair must not be purely imaginary (otherwise, there can be no exponentially decaying tails). Hence the region in which ESs may exist may be delineated by substituting $U, V \sim \exp(\lambda\xi)$ into the linearized equations and solving the resulting eighth-order algebraic equation numerically. It is easy to demonstrate that purely real or imaginary eigenvalues always appear in pairs, and complex eigenvalues in quadruples: if λ is an eigenvalue, then so are $\pm\lambda$ and $\pm\lambda^*$.

We do not display here the full explicit results for the linear spectrum, as they are rather cumbersome. But note that in the quiescent case ($c = 0$) the spectrum is expressible in a closed form [4], and the region in the (χ, D) -plane where ESs may occur is just $|\chi| < 1$, $D > 0$. When $c \neq 0$, the borders to the saddle-centre region of the (c, χ, D) -space retain exactly the same meaning (but there are additional bounding surfaces that are not encountered by any of the ES branches that we have computed). The two degenerate limits of interest are $\chi \rightarrow +1$ (the soliton amplitude going to zero) and $\chi \rightarrow -1$ (smooth transition into a regular soliton).

Eqs. (1) and (2) were numerically solved by means of the same techniques as used in Ref. [4]. That is, posing a two-point boundary-value problem on a long but finite x -interval, with boundary conditions chosen so that to place the solution in the stable or unstable eigenspaces at the endpoints [11]. The boundary-value problem can be posed so that the imaginary parts of $A(\xi)$ and $B(\xi)$ are always even functions, while the real parts are odd. Using these *reversibility* conditions at the midpoint of the soliton, the numerical problem was posed in a more simple way on the half x -interval. Only *fundamental* (single-humped) solitons were sought for, because multi-humped ESs may easily exist but have no chance to be stable [5]. An important ingredient of the technique is the continuation of solutions as various relevant parameters are

varied, which was carried out by means of the known software package AUTO [12].

Quiescent ESs (with $c = 0$) in the present model were found in Ref. [4], which was strongly facilitated by the observation that, at $c = 0$, Eqs. (1) and (2) admit the invariant reduction $V \equiv U^*$, thus reducing the system's order from 8 to 4. The result was that there exist exactly three different branches of quiescent ES solutions. Because ESs exist at isolated values of the energy, each branch can be represented by a curve $E(D)$ in three separate D -intervals (which overlap). Equivalently, the curves can be represented as $D(\chi)$ for $-1 < \chi < 1$.

To the best of our knowledge, moving ESs have never been found before in any model. Our numerical solution of the full system (1) and (2) has demonstrated that a quiescent ESs *cannot* be directly continued into a moving one. Nevertheless, moving ESs exist, but they turn out to be of *codimension two*, i.e., they are *double-isolated*, both in the energy and in the momentum. In other words, a moving ES is described by curves $E(D)$ and $P(D)$. Equivalently, such curves may be represented in (D, c, χ) -space, with the important characteristic of a moving soliton being its velocity c . The mathematical reason for this codimension is that, for the 8th-order model, there are *two* pairs of eigenvalues on the imaginary axis, rather than one pair for the reduced 4th-order model satisfied by the quiescent ESs. A simple count of the dimension of the unstable manifold and that of the symmetric set of the reversibility yields that two parameters are necessary in order to force their intersection and, hence, the existence of a fundamental-type solitary wave.

The results were found to be sensitive to the value of σ (see Eqs. (1) and (2)); note that in the case $c = 0$, σ is trivially scaled out [4]. The case at which most moving ESs were found was $\sigma = 0$. The results obtained for this case are summarized in Fig. 1, which shows that each branch of quiescent ESs solutions gives rise, through a pitchfork bifurcation occurring at some special value of D , to two mutually symmetric branches of moving ESs. In Fig. 1 (and Fig. 2 below), we cut each branch at points where they go over into regular (non-ES) solitons (at $\chi = -1$). Also, we have not depicted the quiescent branches all the way up to $\chi = +1$ due to numerical difficulties in this singular limit. Moreover, it was easy to find additional branches of moving-ES solutions that are *not* connected to the quiescent ones. Only one such disjoint branch is shown in Fig. 1, it persists for all $|\chi| < 1$ without ever bifurcating from a quiescent ES.

Although the case $\sigma = 0$ exactly corresponds to the Thirring model [9], it has no straightforward meaning for optical systems. Therefore, we now focus attention on the most physically relevant case $\sigma = 1/2$. In this case, only one branch of quiescent ESs, corresponding to the smallest values of D , gives rise, through a bifurcation, to branches of moving solitons. Scanning the parameter

space has not yielded any disjoint branch, cf. Fig. 1. This case is shown, in various forms, in Fig. 2. It is interesting, in particular, that the momentum of the moving ESs vanishes at a nonzero value of the velocity, as it passes into the non-embedded region ($\chi < -1$), see Fig. 2b.

The plot that simultaneously shows the energy of the moving ESs and of the coexisting quiescent ESs (Fig. 2c) is especially important. Following the lines of the stability analysis of ESs developed in [5], we can draw some conclusions concerning the stability of both types of the soliton, as well as their possible applications. According to the results of [5], a small perturbation *decreasing* the energy of an isolated ES solution triggers a continual decrease of energy via emission of radiation. In the situation considered in Ref. [5], this would lead to complete decay of ES into radiation. However, in the present case, there is a good chance for a moving ES to shed not only energy, but also momentum, and eventually to turn into a quiescent ES. Because the instability related to this transition is weak (sub-exponential), we may view this as a *tri-stable* system, in which transitions from ESs moving at the velocities $\pm c$ to the quiescent one are possible.

Moreover, this configuration has potential use in an optical-memory device. If an incoming moving ES represents a new bit of information, its radiation-mediated transition into a quiescent ES can be triggered by a specially inserted perturbation (e.g., a localized spatial inhomogeneity, which can be readily made switchable and movable if created by a laser beam focused on a spot in the medium). Thus, the incoming bit could be captured and stored in the memory.

Returning to the general analysis of moving ESs, further numerical explorations have revealed that the single branch of moving ESs existing at $\sigma = 1/2$ is *not* a continuation in σ of any branch existing at $\sigma = 0$; actually, the continuations of all those branches terminate between $\sigma = 0.1$ and $\sigma = 0.2$, and in the same region a new branch appears that, eventually, continues to that found at $\sigma = 1/2$. Continuation of this branch to larger values of σ (which, in principle, may have physical meaning in terms of parallel-coupled dual-core optical fibers or waveguides) shows that it terminates at $\sigma \approx 1.645$. Additional moving ESs exist at still larger values of σ (e.g., at $\sigma = 8.7$), but none was found for $\sigma > 10$.

Finally, one can estimate the values of the physical parameters for possible experimental observation of these ESs in a Bragg-grating medium. A parameter characterizing the relative smallness of the wave (second-derivative) terms in Eqs. (1) and (2), is D/W , W being the ES width. From the data presented in the insets to Figs. 1 and 2, it follows that this parameter takes a nearly constant value, ~ 0.1 , along a moving-ES branch. From the underlying PDEs, it follows that, in terms of physical quantities, the same smallness parameter is $\sim \lambda/4\pi c_0 T$, where $\lambda \equiv 2\pi/k$ is the light wavelength, and T is the temporal width of the pulse. Taking $\lambda \sim 1.5 \mu\text{m}$, and

equating the two expressions for the same smallness, we conclude that one needs $T \sim 10$ fs.

In recently reported experiments in which the temporal solitons were first observed in a Bragg-grating medium, T was very large in comparison with this estimate, ~ 10 ps [13]. However, it is not a problem to create a much shorter pulse. For instance, the first experimental observation of temporal solitons in a second-harmonic-generating medium was carried out using the pulses of the width 58 ps [14]. Generation of the 10 fs pulses is technically complicated but possible. Finally, Fig. 2a clearly shows that the velocity at which moving ESs may be observed includes all values from 0 up to $\sim (1/10)c_0$.

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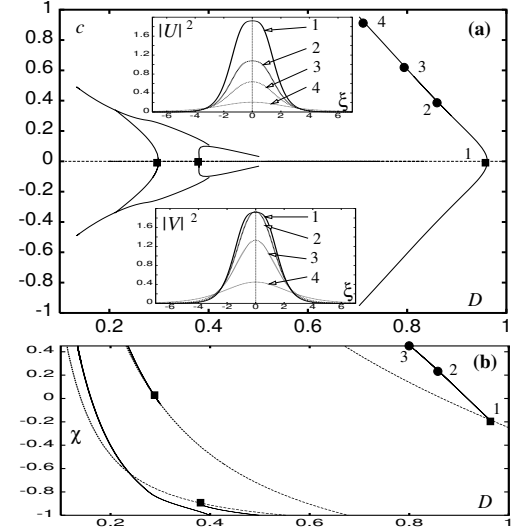


FIG. 1. The solution branches of the quiescent and moving (dotted and solid lines) embedded solitons in the case $\sigma = 0$; squares show bifurcations, circles label points. (a) The velocity vs. the effective dispersion coefficient D . The insets show typical examples of the solutions on the first branch (note the asymmetry between U and V). (b) The effective frequency χ vs. D .

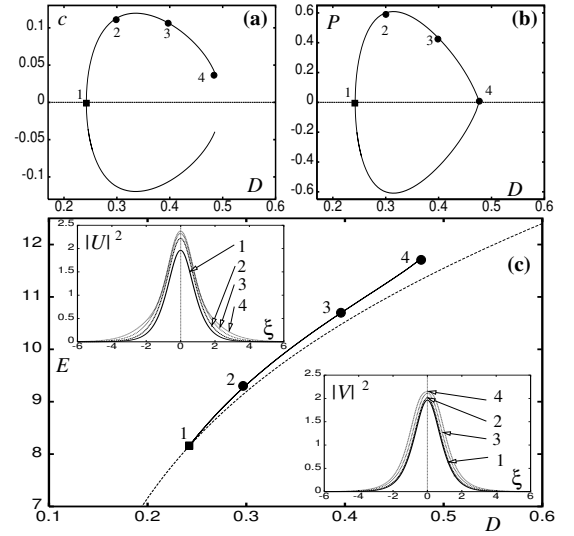


FIG. 2. Various representations of the single branch of the moving embedded solitons existing in the physically relevant case, $\sigma = 1/2$: (a) the same as in Fig. 1; (b) the momentum vs. D ; (c) the energy vs. D , with insets showing the solutions at labeled points. The quiescent solitons branch from which the moving-soliton branch bifurcates is also shown.